

Determine of the wave equation in the task of electrical oscillations

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Abstract: One of the basic equations of mathematical physics (for instance function of two independent variables) is the differential equation with partial derivatives of the second order (3). This equation is called the wave equation, and is provided when considering the process of transverse oscillations of wire, longitudinal oscillations of rod, electrical oscillations in a conductor, torsional vibration at waves, etc... The paper shows how to form the equation (3) which is the equation of motion of each point of wire with abscissa x in time t during its oscillation. It is also shown how to determine the equation (3) in the task of electrical oscillations in a conductor. Then equation (3) is determined, and this solution satisfies the boundary and initial conditions.

1 Formation of a wave equation

In mathematical physics, implied under a wire is a thin elastic thread. Let the wire of the length l at the initial moment match with O from O to l . Assume that the ends of the wire are fixed at points $x=0$ and $x=l$. If the wire is taken from its original position and then released, or without moving the wire, we provide a speed at its initial points, then the wire points will move - the wire starts to oscillate, flickers (oscillates). The task is to determine the shape of the wire at any time and to determine the law of motion of each point of the wire with the abscissa x at the moment t .

When the wire oscillates the deviations of the wire points from the initial position are small, on the Ox axis and are in one plane. The oscillations are determined by the function in $u(x,t)$, which gives the displacement value of the wire points from the abscissa x at the moment t (Fig.1).

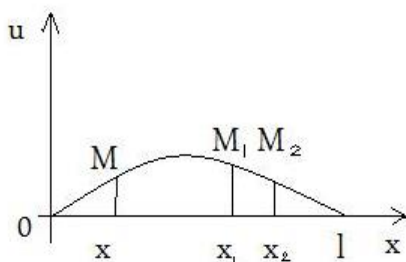


Fig. 1. Displacement value of the wire points

When we observe small wire deviations in the plane (x, u) , we will assume that the length of the wire element is

equal to its projection on the axis Ox , i.e. $M_1M_2 = x_2 - x_1$. We also assume that the tension at all points of the wire is the same and is denoted by T .

Consider the wire element MM' (Fig. 2).

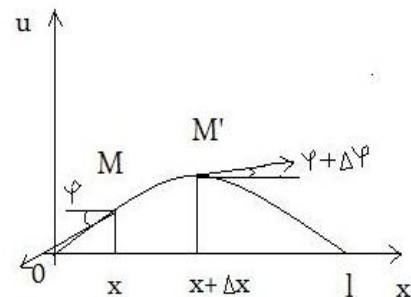


Fig. 2. Wire element MM'

At the ends of this element, at the tangents, the force T acts on the wire. Let the tangents form with Ox axes the angles φ and $\varphi + \Delta\varphi$. Then the projection on the axis of the Ou of the forces acting on the element MM' , will be equal to $T \sin(\varphi + \Delta\varphi) - T \sin \varphi$. When the angle φ is small, we can put that $\text{tg} \varphi \approx \sin \varphi$, so we have:

$$\begin{aligned} T \sin(\varphi + \Delta\varphi) - T \sin \varphi &\approx T \text{tg}(\varphi + \Delta\varphi) - T \text{tg} \varphi = \\ &= T \left[\frac{\partial u(x + \Delta x, t)}{\partial x} - \frac{\partial u(x, t)}{\partial x} \right] = T \frac{\partial^2 u(x + \theta \Delta x, t)}{\partial x^2} \Delta x \approx \\ &\approx T \frac{\partial^2 u(x, t)}{\partial x^2} \Delta x, \quad 0 < \theta < 1 \end{aligned} \quad (1)$$

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Where Lagrange's theorem is applied in the expression in square brackets.

In order to get the equation of motion, it is necessary to equalize the external forces acting on the element with the force of inertia. Let ρ be the specific weight of the wire. Then the mass of the wire element will be $\rho\Delta x$. The element acceleration equals $\partial^2 u / \partial t^2$. According to D'Alembert's principle, we obtain:

$$\rho\Delta x \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2} \Delta x. \quad (2)$$

If we shorten this equation with Δx and put $T/\rho = a^2$, we get the following differential equation with partial derivatives of the second order [1-6]:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad (3)$$

It is a wave equation - the equation of oscillation of a wire. The equation (3) itself is insufficient to fully determine the motion of the wire. The required function in $u(x, t)$ should satisfy even boundary conditions that determine what happens at the ends of the wire $x=0$ and $x=l$, and the initial conditions describing the position of the wire at the initial moment $t=0$.

Let, for example, as we have assumed, the ends of the wire in $x=0$ and $x=l$ be immobile. Then for any t the equalities are exact:

$$u(0, t) = 0, \quad u(l, t) = 0. \quad (4)$$

These equalities are boundary conditions in the task.

In the initial moment $t=0$ the wire has a certain shape, which we have given it. Let this form be determined by the function $f(x)$. Therefore, there must be:

$$u(x, 0) = u|_{t=0} = f(x). \quad (5)$$

Further, the speed needs to be given in the starting point at each point of the wire, which is determined by the function $\varphi(x)$. Accordingly:

$$\frac{\partial u}{\partial t} \Big|_{t=0} = \varphi(x). \quad (6)$$

The conditions (5) and (6) are the initial conditions.

As indicated above, equation (3) is also obtained in the task of electrical oscillations in the conductor. This can be shown in the following way. The electrical current in the conductor is characterized by the size $i(x, t)$ and the tension $v(x, t)$, which depend on the coordinate of the conductor point x and on the time t . Observing the conductor element Δx , we can write that the decrease in the tension on the element Δx equals

$v(x, t) - v(x + \Delta x, t) \approx -\frac{\partial v}{\partial x} \Delta x$. This decrease in

tension is equal to the sum of resistance $iR\Delta x$ and induction $\frac{\partial i}{\partial t} L\Delta x$. Accordingly:

$$-\frac{\partial v}{\partial x} \Delta x = iR\Delta x + \frac{\partial i}{\partial t} L\Delta x, \quad (7)$$

where R and L are coefficients of resistance and induction on the unit of conductor length. The minus sign should be taken because the current flows in the opposite direction from growth of v . By shortening it with Δx , we get the equality:

$$\frac{\partial v}{\partial x} + iR + L \frac{\partial i}{\partial t} = 0. \quad (8)$$

Furthermore, the difference in currents passing through the element Δx in time Δt is:

$$i(x, t) - i(x + \Delta x, t) \approx -\frac{\partial i}{\partial x} \Delta x \Delta t. \quad (9)$$

It is consumed on charging the element, it equals $C\Delta x(\partial v / \partial t)\Delta t$, and on the outflowing through the side surface of the conductor due to the imperfection of the insulation, equaling $Av\Delta x\Delta t$ (here A - is the coefficient of outflow). By equating these expressions and shortening with $\Delta x\Delta t$, we get:

$$\frac{\partial i}{\partial x} + C \frac{\partial v}{\partial t} + Av = 0. \quad (10)$$

The equality (8) and (10) are called *telegraphic equations*.

Equations (8) and (10) can be obtained from the equation system, which contains only the required function $i(x, t)$, and the equations, which contains only the required function $v(x, t)$. Differential equality (10) by x ; equation (8) is differentiated by t , multiplied by S and subtract the two values. We obtain:

$$\frac{\partial^2 i}{\partial x^2} + A \frac{\partial v}{\partial x} - CR \frac{\partial i}{\partial t} - CL \frac{\partial^2 i}{\partial t^2} = 0. \quad (11)$$

From (8) we express $\partial v / \partial x$ and replace it in (11). We get:

$$\frac{\partial^2 i}{\partial x^2} = CL \frac{\partial^2 i}{\partial t^2} + (CR + AL) \frac{\partial i}{\partial t} + A Ri. \quad (12)$$

Analogously, we obtain the equation that determines $v(x, t)$.

Differential equality (8) by x ; equation (10) is differentiated by t , multiplied by L and subtract the two values. We obtain:

$$\frac{\partial^2 v}{\partial x^2} + R \frac{\partial i}{\partial x} - CL \frac{\partial^2 v}{\partial t^2} - AL \frac{\partial v}{\partial t} = 0 \quad (13)$$

From the equation (10) we express $\partial i/\partial x$ and replace it in (13). We get:

$$\frac{\partial^2 v}{\partial x^2} = CL \frac{\partial^2 v}{\partial t^2} + (CR + AL) \frac{\partial v}{\partial t} + ARv. \quad (14)$$

If we can neglect the loss through insulation ($A=0$) and resistance ($R=0$), then the equations (12) and (9) become wave equations:

$$a^2 \frac{\partial^2 i}{\partial x^2} = \frac{\partial^2 i}{\partial t^2}, \quad a^2 \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial t^2}, \quad (15)$$

where: $a^2 = 1/CL$

2 Solving the oscillation (flickering) equation of a wire

A solution to the following equation needs to be found:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \quad (16)$$

satisfying the following conditions:

$$u(0, t) = 0, \quad u(l, t) = 0, \quad (17)$$

$$u(x, 0) = f(x), \quad \frac{\partial u}{\partial t} \Big|_{t=0} = \varphi(x). \quad (18)$$

We will seek the solution of the equation (16), which satisfies the boundary conditions (17) and (18) in the form of products of the two functions $H(h)$ and $T(t)$, the first of which depends on x , and the second on t :

$$u(x, t) = X(x)T(t). \quad (19)$$

By substitution in the relation (10), we obtain:
 $X(x)T''(t) =$. We divide the equation with $a^2 XT$ and we obtain:

$$\frac{T''}{a^2 T} = \frac{X''}{X}. \quad (20)$$

and thus we have separated the variables.

On the left side of the equation is a function, which does not depend on x , and on the right the function which does not depend on t . The equation (20) is only possible in this case when the left and right sides do not depend on either x or t , i.e. they are equal to the constant. Let's denote it with $-\lambda$, where $\lambda > 0$. Thus

$$\frac{T''}{a^2 T} = \frac{X''}{X} = -\lambda. \quad (21)$$

From this equation we obtain two linear differential equations with constant coefficients:

$$X'' + \lambda X = 0 \quad T'' + a^2 \lambda T = 0 \quad (22)$$

The general solutions of these equations are:

$$X(x) = A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x, \quad (23)$$

$$T(t) = C \cos a\sqrt{\lambda} t + D \sin a\sqrt{\lambda} t, \quad (24)$$

where A, B, C, D are the production constants.

By putting the expressions $X(x)$ and $T(t)$ into equality (19), we obtain:

$$u(x, t) = (A \cos \sqrt{\lambda} x + B \sin \sqrt{\lambda} x)(C \cos a\sqrt{\lambda} t + D \sin a\sqrt{\lambda} t). \quad (25)$$

Let's find the constants A and B so that they satisfy the conditions (17) and (18). Since $T(t) \neq 0$ (otherwise it would be $u(x, t) \equiv 0$, which would be contrary to the set condition), then the function $X(x)$ satisfies conditions (17) and (18), i.e. $X(0)=0$, $X(l)=0$. Replacing the values $x=0$ and $x=l$ in equality (23), on the basis of (17) we obtain:

$$0 = A \cdot 1 + B \cdot 0, \quad 0 = A \cos \sqrt{\lambda} l + B \sin \sqrt{\lambda} l. \quad (26)$$

The first equality gives $A=0$. From the second equality follows:

$$B \sin \sqrt{\lambda} l = 0. \quad (27)$$

$B \neq 0$, because otherwise it would be $X \equiv 0$ and $u \equiv 0$, which contradicts the condition. Therefore, it must be $\sin \sqrt{\lambda} l = 0$, where there follows:

$$\sqrt{\lambda} = \frac{n\pi}{l}, \quad (n=1, 2, 3, \dots) \quad (28)$$

(we do not take the value $n=0$, because in this case it would be $X \equiv 0$ and $u \equiv 0$). We obtain:

$$X = B \sin \frac{n\pi}{l} x. \quad (29)$$

The values found λ are called *their own values* for the given boundary task. The corresponding functions $X(x)$ are called *the own functions*.

If instead $-\lambda$ we take $+\lambda = k^2$, the equation (17) would have the form

$$X'' - k^2 X = 0. \quad (30)$$

The general solution of this equation is:

$$X = Ae^{kx} + Be^{-kx}. \quad (31)$$

A solution different from zero in this form cannot satisfy the boundary conditions (11) and (12).

Knowing $\sqrt{\lambda}$, using equality (18), we can write:

$$T(t) = C \cos \frac{an\pi}{l} t + D \sin \frac{an\pi}{l} t \quad (n=1, 2, \dots). \quad (32)$$

For each n , therefore also for each λ , the expressions (29) and (32) are replaced in the equation (19) and we obtain a solution of the equation (3), which satisfies the boundary conditions (17). We designate this solution with $u_n(x, t)$:

$$u_n(x, t) = \sin \frac{n\pi}{l} x (C_n \cos \frac{an\pi}{l} t + D_n \sin \frac{an\pi}{l} t). \quad (33)$$

For each value n we can take different constants C and D and therefore we write C_n and D_n (the constant B is included in C_n and D_n). Since the equation (16) is linear, then the sum of solutions is also a solution, and hence the function represented in the order:

$$u(x, t) = \sum_{n=1}^{\infty} (C_n \cos \frac{an\pi}{l} t + D_n \sin \frac{an\pi}{l} t) \sin \frac{n\pi}{l} x, \quad (34)$$

is also the solution of the differential equation (16), which meets the boundary conditions (17).

The solution (34) needs to satisfy the initial conditions (18). Putting in (34) $t=0$ and taking into account (18), we obtain:

$$f(x) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi}{l} x. \quad (35)$$

If the function $f(x)$ is such that it can be explained in the Fourier series in the interval $(0, l)$, then the condition (35) will be satisfied if we put:

$$C_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi}{l} x dx. \quad (36)$$

Differential equality (34) by t and put that $t=0$.

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} (-C_n \frac{an\pi}{l} \sin \frac{an\pi}{l} t + D_n \frac{an\pi}{l} \cos \frac{an\pi}{l} t) \sin \frac{n\pi}{l} x, \quad t = 0. \quad (37)$$

We obtain:

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} D_n \frac{an\pi}{l} \sin \frac{n\pi}{l} x. \quad (38)$$

From condition (27) we obtain the equality>

$$\varphi(x) = \sum_{n=1}^{\infty} D_n \frac{an\pi}{l} \sin \frac{n\pi}{l} x. \quad (39)$$

In this line Fourier's coefficient is>

$$D_n = \frac{2}{an\pi} \int_0^l \varphi(x) \sin \frac{n\pi}{l} x dx. \quad (40)$$

Different method for solving differential equations are shown in the papers [7-12].

3 Conclusion

The task of electrical oscillations in the conductor is solved. The solution is in the form of Fourier order and satisfies the initial conditions.

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